The antifield Koszul–Tate complex of reducible Noether identities

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A generic degenerate Lagrangian system of even and odd fields is examined in algebraic terms of the Grassmann-graded variational bicomplex. Its Euler–Lagrange operator obeys Noether identities which need not be independent, but satisfy first-stage Noether identities, and so on. We show that, if a certain necessary and sufficient condition holds, one can associate to a degenerate Lagrangian system the exact Koszul–Tate complex with the boundary operator whose nilpotency condition restarts all its Noether and higher-stage Noether identities. This complex provides a sufficient analysis of the degeneracy of a Lagrangian system for the purpose of its BV quantization.

I. INTRODUCTION

As well-known, quantization of a Lagrangian field system essentially depends on the analysis of its degeneracy. One says that a Lagrangian system is degenerate if its Euler–Lagrange operator obeys non-trivial Noether identities. They need not be independent, but satisfy the first-stage Noether identities, which in turn are subject to the second-stage ones, and so on. The hierarchy of reducible Noether identities characterizes the degeneracy of a Lagrangian system in full. Noether's second theorem states the relation between the Noether identities and the gauge symmetries of a Lagrangian system.[1,2] If Noether identities and gauge symmetries are finitely generated, they are parameterized by the modules of

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antifields and ghosts, respectively. An original Lagrangian is extended to these antifields and ghosts in order to satisfy the so-called master equation. This extended Lagrangian is the starting point of the Batalin-Vilkovisky (BV) quantization of a degenerate Lagrangian field system.[3,4]

Let us note that the notion of a reducible Noether identity has come from that of a reducible constraint. Their Koszul–Tate complex has been invented by analogy with that of constraints[5] under a rather restrictive regularity condition that field equations as well as Noether identities of arbitrary stage can be locally separated into the independent and dependent ones.[6,7] This condition also comes from the case of a constraint locally given by a finite number of functions to which the inverse mapping theorem can be applied. In contrast with constraints, Noether and higher-stage Noether identities are differential operators. They are locally given by a set of functions and their jet prolongations on an infinite order jet manifold. Since the latter is a Fréchet, but not Banach manifold, the inverse mapping theorem fails to be valid. Here, we follow the general definition of Noether identities of differential operators.[8] This definition reproduces that in Refs. [1,2] if Noether identities are finitely generated. Their Koszul–Tate complex is constructed iff a certain homology regularity condition holds.

Our goal is the following. Bearing in mind BV quantization, we address a generic Lagrangian systems of even and odd fields on an arbitrary smooth manifold X ($\dim X = n$). It is algebraically described in terms of a certain bigraded differential algebra (henceforth BGDA) $\mathcal{S}^*_\infty[F;Y]$ which is split into the Grassmann-graded variational bicomplex, generalizing the variational bicomplex on fiber bundles (Section II). If a fiber bundle $Y \to X$ of even fields is affine, this algebra has been defined as the product of graded algebras of odd and even fields.[2,9] Here, its definition is generalized to an arbitrary fiber bundle $Y \to X$. In this case, elements of $\mathcal{S}^*_\infty[F;Y]$ are Grassmann-graded differential forms on the infinite order jet manifold $J^\infty Y$ of sections of $Y \to X$, but not on X. Let $L \in \mathcal{S}^{0,n}_\infty[F;Y]$ be a Lagrangian and $\delta L \in \mathcal{S}^{1,n}_\infty[F;Y]$ its Euler–Lagrange operator. We associate to δL the chain complex (13) whose boundaries vanish on-shell, i.e., on $\mathrm{Ker}\,\delta L$ (Proposition 4). It is a complex of a certain $C^\infty(X)$-module $\mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]$ of Grassmann-graded densities on the infinite order jet manifold $J^\infty Y$. For our purpose, this complex can be replaced with the short zero-exact complex $\mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]_{\leq 2}$ (14).

Remark 1: If there is no danger of confusion, elements of homology are identified to its representatives. A chain complex is called r-exact if its homology of $k \leq r$ is trivial.

The Noether identities of the Euler–Lagrange operator δL are defined as nontrivial elements of the first homology $H_1(\overline{\delta})$ of the complex (14) (Definition 5). Let this homology be finitely generated by a projective graded $C^\infty(X)$-module of finite rank. In accordance with the Serre–Swan theorem generalized to graded manifolds (Theorem 1), one can introduce the corresponding module of antifields and extend the complex (14) to the one-exact

complex $\mathcal{P}^{0,n}_\infty[\overline{E}^*\overline{Y}^*;F;Y;\overline{F}^*\overline{V}^*]_{\leq 3}$ (22) with the boundary operator δ_0 (21) whose nilpotency conditions are equivalent to the above-mentioned Noether identities (Proposition 6). First-stage Noether identities are defined as two-cycles of this complex. They are trivial if two-cycles are boundaries, but the converse need not be true. Trivial first-stage Noether identities are boundaries iff a certain homology condition (called the two-homology regularity condition) holds (Proposition 8). In this case, the first-stage Noether identities are identified to nontrivial elements of the second homology of the complex (22). If this homology is finitely generated, the complex (22) is extended to the two-exact complex $\mathcal{P}^{0,n}_\infty[\overline{E}_1^*\overline{E}^*\overline{Y}^*;F;Y;\overline{F}^*\overline{V}^*\overline{V}_1^*]_{\leq 4}$ (33) with the boundary operator δ_1 (32) whose nilpotency conditions are equivalent to the Noether and first-stage Noether identities (Proposition 10). If the third homology of this complex is not trivial, the second-stage Noether identities exist, and so on. Iterating the arguments, we come to the following.

Let we have the $(N+1)$-exact complex $\mathcal{P}^{0,n}_\infty\{N\}_{\leq N+3}$ (37) such that: (i) the nilpotency conditions of its boundary operator δ_N (35) reproduce Noether and k-stage Noether identities for $k\leq N$, (ii) the $(N+1)$-homology regularity condition holds. This condition states that any $\delta_{k<N-1}$-cycle $\phi\in\mathcal{P}^{0,n}_\infty\{k\}_{k+3}$ is a δ_{k+1}-boundary (Definition 11). Then the $(N+1)$-stage Noether identities are defined as $(N+2)$-cycles of this complex. They are trivial if cycles are boundaries, while the converse is true iff the $(N+2)$-homology regularity condition is satisfied. In this case, $(N+1)$-stage Noether identities are identified to nontrivial elements of the $(N+2)$-homology of the complex (37) (item (i) of Theorem 13). Let this homology is finitely generated. By means of antifields, this complex is extended to the $(N+2)$-exact complex $\mathcal{P}_\infty\{N+1\}_{\leq N+4}$ (45) with the boundary operator δ_{N+1} (44) whose nilpotency restarts all the Noether identities up to stage $(N+1)$ (item (ii) of Theorem 13).

This iteration procedure results in the exact Koszul–Tate complex of antifields with the boundary operator whose nilpotency conditions reproduce all Noether and higher Noether identities characterizing the degeneracy of a differential operator δL.

In Section V, we address the particular variant of topological BF theory with the Lagrangian (47) for a scalar A and $(n-1)$-form B as an example of a reducible degenerate Lagrangian system[1] where the homology regularity condition is verified (Lemma 14), Noether and k-stage Noether identities are proved to be finitely generated, and its Koszul–Tate complex (62) is constructed.

Remark 2: Throughout the paper, smooth manifolds are assumed to be real, finite-dimensional, Hausdorff, second-countable (consequently, paracompact) and connected. By a Grassmann algebra over a ring $\mathcal{K}$ is meant a $\mathbb{Z}_2$-graded exterior algebra of some $\mathcal{K}$-module. We restrict our consideration to graded manifolds $(Z,\mathfrak{A})$ with structure sheaves $\mathfrak{A}$ of Grassmann algebras of finite rank.[10,11] The symbols $|.|$ and $[.]$ stand for the form degree and Grassmann parity, respectively. We denote by Λ, Σ, Ξ, Ω the symmetric multi-indices, e.g., $\Lambda=(\lambda_1...\lambda_k)$, $\lambda+\Lambda=(\lambda\lambda_1...\lambda_k)$. Summation over a multi-index $\Lambda=(\lambda_1...\lambda_k)$

throughout means separate summation over each its index λ_i.

II. GRASSMANN-GRADED LAGRANGIAN SYSTEMS

Let $Y \to X$ be a fiber bundle and J^rY the jet manifolds of its sections. They form the inverse system

$$X \overset{\pi}{\longleftarrow} Y \overset{\pi^1_0}{\longleftarrow} J^1Y \longleftarrow \cdots J^{r-1}Y \overset{\pi^r_{r-1}}{\longleftarrow} J^rY \longleftarrow \cdots, \tag{1}$$

where π^r_{r-1} are affine bundles, and $r = 0$ conventionally stands for Y. Its projective limit $(J^\infty Y; \pi^\infty_r : J^\infty Y \to J^rY)$ is a paracompact Fréchet manifold. A bundle atlas $\{(U_Y; x^\lambda, y^i)\}$ of $Y \to X$ induces the coordinate atlas

$$\begin{aligned} &\{((\pi^\infty_0)^{-1}(U_Y); x^\lambda, y^i_\Lambda)\}, \qquad y'^i_{\lambda+\Lambda} = \frac{\partial x^\mu}{\partial x'^\lambda} d_\mu y'^i_\Lambda, \qquad 0 \leq |\Lambda|, \\ &d_\lambda = \partial_\lambda + \sum_{0\leq|\Lambda|} y^i_{\lambda+\Lambda}\partial^\Lambda_i, \qquad d_\Lambda = d_{\lambda_1} \circ \cdots \circ d_{\lambda_k}, \end{aligned} \tag{2}$$

of $J^\infty Y$, where d_λ are total derivatives. We further assume that the cover $\{\pi(U_Y)\}$ of X is also the cover of atlases of all fiber bundles over X in question. The inverse system (1) yields the direct system

$$\mathcal{O}^*X \xrightarrow{\pi^*} \mathcal{O}^*Y \xrightarrow{\pi^{1\,*}_0} \mathcal{O}^*_1Y \longrightarrow \cdots \mathcal{O}^*_{r-1}Y \xrightarrow{\pi^r_{r-1}{}^*} \mathcal{O}^*_rY \longrightarrow \cdots \tag{3}$$

of algebras $\mathcal{O}^*_rY$ of exterior forms on jet manifolds J^rY with respect to the pull-back monomorphisms $\pi^r_{r-1}{}^*$. Its direct limit is the graded differential algebra (henceforth GDA) $\mathcal{O}^*_\infty Y$ of all exterior forms on finite order jet manifolds modulo the pull-back identification.

Let us extend the GDA $\mathcal{O}^*_\infty Y$ to graded forms on graded manifolds whose bodies are jet manifolds J^rY of Y.[2,9] Note that there are different approaches to treat odd fields on a smooth manifold X, but the following variant of the Serre–Swan theorem motivates us to describe them in terms of graded manifolds whose body is X.

Theorem 1: Let Z be a smooth manifold. A Grassmann algebra $\mathcal{A}$ over the ring $C^\infty(Z)$ of smooth real functions on Z is isomorphic to the Grassmann algebra of graded functions on a graded manifold with a body Z iff it is the exterior algebra of some projective $C^\infty(Z)$-module of finite rank.

Proof: The proof follows at once from the Batchelor theorem[10] and the Serre-Swan theorem generalized to an arbitrary smooth manifold.[11,12] The Batchelor theorem states that any graded manifold $(Z, \mathfrak{A})$ with a body Z is isomorphic to the one $(Z, \mathfrak{A}_Q)$ with the structure sheaf $\mathfrak{A}_Q$ of germs of sections of the exterior bundle

$$\wedge Q^* = \mathbb{R} \underset{Z}{\oplus} Q^* \underset{Z}{\oplus} \overset{2}{\wedge} Q^* \underset{Z}{\oplus} \cdots,$$

where Q^* is the dual of some vector bundle $Q \to Z$. Let us call $(Z, \mathfrak{A}_Q)$ the simple graded manifold with the structure vector bundle Q. Its ring $\mathcal{A}_Q$ of graded functions (sections of $\mathfrak{A}_Q$) is the $\mathbb{Z}_2$-graded exterior algebra of the $C^\infty(Z)$-module of sections of $\wedge Q^* \to Z$. By virtue of the Serre–Swan theorem, a $C^\infty(Z)$-module is isomorphic to the module of sections of a smooth vector bundle over Z iff it is a projective module of finite rank.

In field models, Batchelor's isomorphism is usually fixed from the beginning. Therefore, we further consider simple graded manifolds $(Z, \mathfrak{A}_Q)$. One associates to $(Z, \mathfrak{A}_Q)$ the following BGDA $\mathcal{S}^*[Q; Z]$.[10] Let $\mathfrak{d}\mathfrak{A}_Q$ be the sheaf of graded derivations of $\mathfrak{A}_Q$. Its global sections make up the real Lie superalgebra $\mathfrak{d}\mathcal{A}_Q$ of graded derivations of the $\mathbb{R}$-ring $\mathcal{A}_Q$. Then the Chevalley–Eilenberg complex of $\mathfrak{d}\mathcal{A}_Q$ with coefficients in $\mathcal{A}_Q$ can be constructed.[13] Its subcomplex $\mathcal{S}^*[Q; Z]$ of $\mathcal{A}_Q$-linear morphisms is the Grassmann-graded Chevalley–Eilenberg differential calculus

$$0 \to \mathbb{R} \to \mathcal{A}_Q \xrightarrow{d} \mathcal{S}^1[Q; Z] \xrightarrow{d} \cdots \mathcal{S}^k[Q; Z] \xrightarrow{d} \cdots$$

over a $\mathbb{Z}_2$-graded commutative $\mathbb{R}$-ring $\mathcal{A}_Q$. The graded exterior product $\wedge$ and Chevalley–Eilenberg coboundary operator d (the graded exterior differential) make $\mathcal{S}^*[Q; Z]$ into a BGDA whose elements obey the relations

$$\phi \wedge \phi' = (-1)^{|\phi||\phi'| + [\phi][\phi']} \phi' \wedge \phi, \qquad d(\phi \wedge \phi') = d\phi \wedge \phi' + (-1)^{|\phi|} \phi \wedge d\phi'.$$

Given the GDA $\mathcal{O}^*Z$ of exterior forms on Z, there are the monomorphism $\mathcal{O}^*Z \to \mathcal{S}^*[Q; Z]$ and the body epimorphism $\mathcal{S}^*[Q; Z] \to \mathcal{O}^*Z$. The following facts are essential.[9,11]

Lemma 2: The BGDA $\mathcal{S}^*[Q; Z]$ is a minimal differential calculus over $\mathcal{A}_Q$, i.e., it is generated by elements df, $f \in \mathcal{A}_Q$.

Lemma 3: Given a ring R, let $\mathcal{K}$, $\mathcal{K}'$ be R-rings and $\mathcal{A}$, $\mathcal{A}'$ the Grassmann algebras over $\mathcal{K}$ and $\mathcal{K}'$, respectively. Then a homomorphism (resp. a monomorphism) $\rho : \mathcal{A} \to \mathcal{A}'$ yields a homomorphism (resp. a monomorphism) of the minimal Chevalley–Eilenberg differential calculus over a $\mathbb{Z}_2$-graded R-ring $\mathcal{A}$ to that over $\mathcal{A}'$ given by the map $da \mapsto d(\rho(a))$, $a \in \mathcal{A}$.

One can think of elements of the BGDA $\mathcal{S}^*[Q; Z]$ as being graded exterior forms on Z as follows. Given an open subset $U \subset Z$, let $\mathcal{A}_U$ be the Grassmann algebra of sections of the sheaf $\mathfrak{A}_Q$ over U, and let $\mathcal{S}^*[Q; U]$ be the Chevalley–Eilenberg differential calculus over $\mathcal{A}_U$. Given an open set $U' \subset U$, the restriction morphisms $\mathcal{A}_U \to \mathcal{A}_{U'}$ yield a homomorphism of BGDAs $\mathcal{S}^*[Q; U] \to \mathcal{S}^*[Q; U']$. Thus, we obtain the presheaf $\{U, \mathcal{S}^*[Q; U]\}$ of BGDAs on a manifold Z and the sheaf $\mathfrak{S}^*[Q; Z]$ of BGDAs of germs of this presheaf. Since $\{U, \mathcal{A}_U\}$ is the canonical presheaf of $\mathfrak{A}_Q$, the canonical presheaf of $\mathfrak{S}^*[Q; Z]$ is $\{U, \mathcal{S}^*[Q; U]\}$. In particular, $\mathcal{S}^*[Q; Z]$ is the BGDA of global sections of the sheaf $\mathfrak{S}^*[Q; Z]$, and there is the restriction morphism $\mathcal{S}^*[Q; Z] \to \mathcal{S}^*[Q; U]$ for any open $U \subset Z$. Due to this morphism, elements of $\mathcal{S}^*[Q; Z]$ can be written in the following local form.

Given bundle coordinates (z^A, q^a) on Q and the corresponding fiber basis $\{c^a\}$ for $Q^* \to X$, the tuple (z^A, c^a) is called a local basis for the graded manifold $(Z, \mathfrak{A}_Q)$.[9] With respect to this basis, graded functions read

$$f = \sum_{k=0} \frac{1}{k!} f_{a_1 \ldots a_k} c^{a_1} \cdots c^{a_k}, \qquad f \in C^\infty(Z), \tag{4}$$

where we omit the symbol of the exterior product of elements c^a. Due to the canonical vertical splitting $VQ = Q \times Q$, the fiber basis $\{\partial_a\}$ for the vertical tangent bundle $VQ \to Q$ of $Q \to Z$ is the dual of $\{c^a\}$. Then graded derivations take the local form $u = u^A \partial_A + u^a \partial_a$, where u^A, u^a are local graded functions. They act on graded functions (4) by the rule

$$u(f_{a\ldots b} c^a \cdots c^b) = u^A \partial_A (f_{a\ldots b}) c^a \cdots c^b + u^d f_{a\ldots b} \partial_d \rfloor (c^a \cdots c^b). \tag{5}$$

Relative to the dual local bases $\{dz^A\}$ for T^*Z and $\{dc^b\}$ for Q^*, graded one-forms read $\phi = \phi_A dz^A + \phi_a dc^a$. The duality morphism is given by the interior product

$$u \rfloor \phi = u^A \phi_A + (-1)^{[\phi_a]} u^a \phi_a, \qquad u \in \mathfrak{d}\mathcal{A}_Q, \qquad \phi \in \mathcal{S}^1[Q; Z].$$

The graded exterior differential reads

$$d\phi = dz^A \wedge \partial_A \phi + dc^a \wedge \partial_a \phi,$$

where the derivations ∂_A and ∂_a act on coefficients of graded exterior forms by the formula (5), and they are graded commutative with the graded exterior forms dz^A and dc^a.

We define jets of odd fields as simple graded manifolds modelled over jet bundles over X.[2,9] This definition differs from the definition of jets of a graded commutative ring[11] and that of jets of a graded fiber bundle,[14] but reproduces the heuristic notion of jets of odd ghosts in Lagrangian BRST theory.[7,15]

Given a vector bundle $F \to X$, let us consider the simple graded manifold $(J^rY, \mathfrak{A}_{F_r})$ whose body is J^rY and the structure bundle is the pull-back

$$F_r = J^rY \underset{X}{\times} J^rF$$

onto J^rY of the jet bundle $J^rF \to X$, which is also a vector bundle. Given the simple graded manifold $(J^{r+1}Y, \mathfrak{A}_{F_{r+1}})$, there is an epimorphism of graded manifolds

$$(J^{r+1}Y, \mathfrak{A}_{F_{r+1}}) \to (J^rY, \mathfrak{A}_{F_r}).$$

It consists of the open surjection π^{r+1}_r and the sheaf monomorphism $\pi^{r+1*}_r \mathfrak{A}_{F_r} \to \mathfrak{A}_{F_{r+1}}$, where $\pi^{r+1*}_r \mathfrak{A}_{F_r}$ is the pull-back onto $J^{r+1}Y$ of the topological fiber bundle $\mathfrak{A}_{F_r} \to J^rY$. This sheaf monomorphism induces the monomorphism of the canonical presheaves $\overline{\mathfrak{A}}_{F_r} \to \overline{\mathfrak{A}}_{F_{r+1}}$,

which associates to each open subset $U\subset J^{r+1}Y$ the ring of sections of $\mathfrak{A}_{F_r}$ over $\pi^{r+1}_r(U)$. Accordingly, there is the monomorphism of $\mathbb{Z}_2$-graded rings $\mathcal{A}_{F_r}\to\mathcal{A}_{F_{r+1}}$. By virtue of Lemmas 2 and 3, this monomorphism yields the monomorphism of BGDAs

$$\mathcal{S}^*[F_r;J^rY]\to\mathcal{S}^*[F_{r+1};J^{r+1}Y]. \tag{6}$$

As a consequence, we have the direct system of BGDAs

$$\mathcal{S}^*[Y\underset{X}{\times}F;Y]\longrightarrow\mathcal{S}^*[F_1;J^1Y]\longrightarrow\cdots\mathcal{S}^*[F_r;J^rY]\longrightarrow\cdots, \tag{7}$$

whose direct limit $\mathcal{S}^*_\infty[F;Y]$ is a BGDA of all graded differential forms $\phi\in\mathcal{S}^*[F_r;J^rY]$ on jet manifolds J^rY modulo monomorphisms (6). The monomorphisms $\mathcal{O}^*_rY\to\mathcal{S}^*[F_r;J^rY]$ provide the monomorphism $\mathcal{O}^*_\infty Y\to\mathcal{S}^*_\infty[F;Y]$ of their direct limits. In particular, $\mathcal{S}^*_\infty[F;Y]$ is an $\mathcal{O}^0_\infty Y$-algebra. Accordingly, the body epimorphisms $\mathcal{S}^*[F_r;J^rY]\to\mathcal{O}^*_rY$ yield the epimorphism of $\mathcal{O}^0_\infty Y$-modules $\mathcal{S}^*_\infty[F;Y]\to\mathcal{O}^*_\infty Y$.

If $Y\to X$ is an affine bundle, we recover the BGDA introduced in Refs. [2, 9] by restricting the ring $\mathcal{O}^0_\infty Y$ to its subring $\mathcal{P}^0_\infty Y$ of polynomial functions, but now elements of $\mathcal{S}^*_\infty[F;Y]$ are graded exterior forms on $J^\infty Y$. Indeed, let $\mathfrak{S}^*[F_r;J^rY]$ be the sheaf of BGDAs on J^rY and $\overline{\mathfrak{S}}^*[F_r;J^rY]$ its canonical presheaf whose elements are the Chevalley–Eilenberg differential calculus over elements of the presheaf $\overline{\mathfrak{A}}_{F_r}$. Then the presheaf monomorphisms $\overline{\mathfrak{A}}_{F_r}\to\overline{\mathfrak{A}}_{F_{r+1}}$ yield the direct system of presheaves

$$\overline{\mathfrak{S}}^*[Y\times F;Y]\longrightarrow\overline{\mathfrak{S}}^*[F_1;J^1Y]\longrightarrow\cdots\overline{\mathfrak{S}}^*[F_r;J^rY]\longrightarrow\cdots, \tag{8}$$

whose direct limit $\overline{\mathfrak{S}}^*_\infty[F;Y]$ is a presheaf of BGDAs on the infinite order jet manifold $J^\infty Y$. Let $\mathfrak{T}^*_\infty[F;Y]$ be the sheaf of BGDAs of germs of the presheaf $\overline{\mathfrak{S}}^*_\infty[F;Y]$. The structure module $\Gamma(\mathfrak{T}^*_\infty[F;Y])$ of sections of $\mathfrak{T}^*_\infty[F;Y]$ is a BGDA such that, given an element $\phi\in\Gamma(\mathfrak{T}^*_\infty[F;Y])$ and a point $z\in J^\infty Y$, there exist an open neighbourhood U of z and a graded exterior form $\phi^{(k)}$ on some finite order jet manifold J^kY so that $\phi|_U=\pi^{\infty*}_k\phi^{(k)}|_U$. In particular, there is the monomorphism $\mathcal{S}^*_\infty[F;Y]\to\Gamma(\mathfrak{T}^*_\infty[F;Y])$.

Due to this monomorphism, one can restrict $\mathcal{S}^*_\infty[F;Y]$ to the coordinate chart (2) and say that $\mathcal{S}^*_\infty[F;Y]$ as an $\mathcal{O}^0_\infty Y$-algebra is locally generated by the elements

$$(1,c^a_\Lambda,dx^\lambda,\theta^a_\Lambda=dc^a_\Lambda-c^a_{\lambda+\Lambda}dx^\lambda,\theta^i_\Lambda=dy^i_\Lambda-y^i_{\lambda+\Lambda}dx^\lambda),\qquad 0\leq|\Lambda|.$$

We agree to call (y^i,c^a) the local basis for $\mathcal{S}^*_\infty[F;Y]$. Let the collective symbol s^A stand for its elements. Accordingly, the notation s^A_Λ and $\theta^A_\Lambda=ds^A_\Lambda-s^A_{\lambda+\Lambda}dx^\lambda$ is introduced. For the sake of simplicity, we further denote $[A]=[s^A]$.

The BGDA $\mathcal{S}^*_\infty[F;Y]$ is decomposed into $\mathcal{S}^0_\infty[F;Y]$-modules $\mathcal{S}^{k,r}_\infty[F;Y]$ of k-contact and r-horizontal graded forms. Accordingly, the graded exterior differential d on $\mathcal{S}^*_\infty[F;Y]$ falls into the sum $d=d_H+d_V$ of the total and vertical differentials, where

$$d_H(\phi)=dx^\lambda\wedge d_\lambda(\phi),\qquad d_\lambda=\partial_\lambda+\sum_{0\leq|\Lambda|}s^A_{\lambda+\Lambda}\partial^\Lambda_A.$$

Given the projector

$$\varrho=\sum_{k>0}\frac{1}{k}\overline{\varrho}\circ h_k\circ h^n, \qquad \overline{\varrho}(\phi)=\sum_{0\leq|\Lambda|}(-1)^{|\Lambda|}\theta^A\wedge[d_\Lambda(\partial^\Lambda_A\rfloor\phi)], \qquad \phi\in\mathcal{S}^{>0,n}_\infty[F;Y],$$

and the graded variational operator $\delta=\varrho\circ d$, the BGDA $\mathcal{S}^*_\infty[F;Y]$ is split into the above mentioned Grassmann-graded variational bicomplex.[7,8] We restrict our consideration to its short variational subcomplex

$$0\longrightarrow\mathbb{R}\longrightarrow\mathcal{S}^0_\infty[F;Y]\xrightarrow{d_H}\mathcal{S}^{0,1}_\infty[F;Y]\cdots\xrightarrow{d_H}\mathcal{S}^{0,n}_\infty[F;Y]\xrightarrow{\delta}\mathbf{E}_1, \quad \mathbf{E}_1=\varrho(\mathcal{S}^{1,n}_\infty[F;Y]).$$

One can think of its even elements

$$\begin{aligned}
&L=\mathcal{L}\omega\in\mathcal{S}^{0,n}_\infty[F;Y], \qquad \omega=dx^1\wedge\cdots\wedge dx^n,\\
&\delta L=\theta^A\wedge\mathcal{E}_A\omega=\sum_{0\leq|\Lambda|}(-1)^{|\Lambda|}\theta^A\wedge d_\Lambda(\partial^\Lambda_A L)\omega\in\mathbf{E}_1
\end{aligned} \tag{9}$$

as being a graded Lagrangian and its Euler–Lagrange operator. A pair $(\mathcal{S}^*_\infty[F;Y],L)$ is further called a graded Lagrangian system.

Let $\vartheta\in\mathfrak{d}\mathcal{S}^0_\infty[F;Y]$ be a graded derivation of the $\mathbb{R}$-ring $\mathcal{S}^0_\infty[F;Y]$.[2,9] The interior product $\vartheta\rfloor\phi$ and the Lie derivative $\mathbf{L}_\vartheta\phi$, $\phi\in\mathcal{S}^*_\infty[F;Y]$, are defined by the formulae

$$\begin{aligned}
&\vartheta\rfloor\phi=\vartheta^\lambda\phi_\lambda+(-1)^{[\phi_A]}\vartheta^A\phi_A, \qquad \phi\in\mathcal{S}^1_\infty[F;Y],\\
&\vartheta\rfloor(\phi\wedge\sigma)=(\vartheta\rfloor\phi)\wedge\sigma+(-1)^{|\phi|+[\phi][\vartheta]}\phi\wedge(\vartheta\rfloor\sigma), \qquad \phi,\sigma\in\mathcal{S}^*_\infty[F;Y],\\
&\mathbf{L}_\vartheta\phi=\vartheta\rfloor d\phi+d(\vartheta\rfloor\phi), \qquad \mathbf{L}_\vartheta(\phi\wedge\sigma)=\mathbf{L}_\vartheta(\phi)\wedge\sigma+(-1)^{[\vartheta][\phi]}\phi\wedge\mathbf{L}_\vartheta(\sigma).
\end{aligned}$$

A graded derivation ϑ is said to be contact if the Lie derivative $\mathbf{L}_\vartheta$ preserves the ideal of contact graded forms of the BGDA $\mathcal{S}^*_\infty[F;Y]$. With respect to the local basis $\{s^A\}$ for the BGDA $\mathcal{S}^*_\infty[F;Y]$, any contact graded derivation takes the form

$$\vartheta=\vartheta_H+\vartheta_V=\vartheta^\lambda d_\lambda+(\vartheta^A\partial_A+\sum_{0<|\Lambda|}d_\Lambda\vartheta^A\partial^\Lambda_A),$$

where the tuple of graded derivations $\{\partial_\lambda,\partial^\Lambda_A\}$ is the dual of the tuple $\{dx^\lambda,ds^A_\Lambda\}$ of generating elements of the $\mathcal{S}^0_\infty[F;Y]$-algebra $\mathcal{S}^*_\infty[F;Y]$, and ϑ^λ, ϑ^A are local graded functions.

We restrict our consideration to vertical contact graded derivations

$$\vartheta=\sum_{0\leq|\Lambda|}d_\Lambda\upsilon^A\partial^\Lambda_A. \tag{10}$$

Such a derivation is completely determined by its first summand

$$\upsilon=\upsilon^A(x^\lambda,s^A_\Lambda)\partial_A, \qquad 0\leq|\Lambda|\leq k, \tag{11}$$

called a generalized graded vector field. It is said to be nilpotent if

$$\mathbf{L}_\vartheta(\mathbf{L}_\vartheta\phi)=\sum_{0\leq|\Sigma|,0\leq|\Lambda|}(\upsilon^B_\Sigma\partial^\Sigma_B(\upsilon^A_\Lambda)\partial^\Lambda_A+(-1)^{[B][\upsilon^A]}\upsilon^B_\Sigma\upsilon^A_\Lambda\partial^\Sigma_B\partial^\Lambda_A)\phi=0$$

for any horizontal graded form $\phi\in\mathcal{S}^{0,*}_\infty[F;Y]$. One can show that ϑ (10) is nilpotent only if it is odd and iff all υ^A obey the equality

$$\vartheta(\upsilon^A)=\sum_{0\leq|\Sigma|}\upsilon^B_\Sigma\partial^\Sigma_B(\upsilon^A)=0. \tag{12}$$

For the sake of simplicity, the common symbol further stands for a generalized vector field (11), the contact graded derivation (10) determined by this field and the Lie derivative $\mathbf{L}_\vartheta$. We agree to call all these operators the graded derivation of the BGDA $\mathcal{S}^*_\infty[F;Y]$.

III. NOETHER IDENTITIES IN A GENERAL SETTING

Given a graded Lagrangian system $(\mathcal{S}^*_\infty[F;Y],L)$, let us construct the manifested Koszul–Tate complex of its Noether identities.

The main ingredient in this construction is BGDAs of the following type. Given a vector bundle $E\to X$, let us consider the BGDA $\mathcal{S}^*_\infty[F;E_Y]$, where E_Y denotes the pull-back of E onto Y. There are monomorphisms of $\mathcal{O}^0_\infty Y$-algebras

$$\mathcal{S}^*_\infty[F;Y]\to\mathcal{S}^*_\infty[F;E_Y],\qquad \mathcal{O}^*_\infty E\to\mathcal{S}^*_\infty[F;E_Y],$$

whose images contain the common subalgebra $\mathcal{O}^*_\infty Y$. Let us consider: (i) the subring $\mathcal{P}^0_\infty E_Y\subset\mathcal{O}^0_\infty E_Y$ of polynomial functions in fiber coordinates of the vector bundles $J^rE_Y\to J^rY$, $r\in\mathbb{N}$, (ii) the corresponding subring $\mathcal{P}^0_\infty[F;E_Y]\subset\mathcal{S}^0_\infty[F;E_Y]$ of graded functions with polynomial coefficients belonging to $\mathcal{P}^0_\infty E_Y$, (iii) the subalgebra $\mathcal{P}^*_\infty[F;Y;E]$ of the BGDA $\mathcal{S}^*_\infty[F;E_Y]$ over the subring $\mathcal{P}^0_\infty[F;E_Y]$. Given vector bundles V,V',E,E' over X, we further use the notation

$$\mathcal{P}^*_\infty[V'V;F;Y;EE']=\mathcal{P}^*_\infty[V'\underset{X}{\times}V\underset{X}{\times}F;Y;E\underset{X}{\times}E'].$$

By a density-dual of a vector bundle $E\to X$ is meant

$$\overline{E}^*=E^*\underset{X}{\otimes}\overset{n}{\wedge}T^*X.$$

For the sake of simplicity, we restrict our consideration to Lagrangian systems where a fiber bundle $Y\to X$ of even fields admits the vertical splitting $VY=Y\times W$, where W is a vector bundle over X. This is case of almost all field models. In a general setting, one must

require that transition functions of fiber bundles over Y under consideration do not vanish on-shell. Let $\overline{Y}^*$ denote the density-dual of W in the above mentioned vertical splitting.

Proposition 4: One can associate to a graded Lagrangian system $(\mathcal{S}^*_\infty[F;Y],L)$, a chain complex whose boundaries vanish on shell (see the complex (13) below).

Proof: Let us extend the BGDA $\mathcal{S}^*_\infty[F;Y]$ to the BGDA $\mathcal{P}^*_\infty[\overline{Y}^*;F;Y;\overline{F}^*]$ whose local basis is $\{s^A,\overline{s}_A\}$, where $[\overline{s}_A]=([A]+1)\operatorname{mod}2$. Following the terminology of Lagrangian BRST theory,[2,5] we call $\overline{s}_A$ the antifields of antifield number 1. The BGDA $\mathcal{P}^0_\infty[\overline{Y}^*;F;Y;\overline{F}^*]$ is provided with the nilpotent graded derivation $\overline{\delta}=\overleftarrow{\partial}^A\mathcal{E}_A$, where $\mathcal{E}_A$ are the graded variational derivatives (9) and the tuple of graded right derivations $\{\overleftarrow{\partial}^{\Lambda A}\}$ is the dual of the tuple of contact graded forms $\{\theta_{\Lambda A}\}$. Because of the expression (9) for δL, it is convenient to deal with a graded derivation $\overline{\delta}$ acting on graded functions and forms ϕ on the right by the rule

$$\overline{\delta}(\phi)=d\phi\lfloor\overline{\delta}+d(\phi\lfloor\overline{\delta}), \qquad \overline{\delta}(\phi\wedge\phi')=(-1)^{[\phi']}\overline{\delta}(\phi)\wedge\phi'+\phi\wedge\overline{\delta}(\phi').$$

We call $\overline{\delta}$ the Koszul–Tate differential. Let us consider the module $\mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]$ of graded densities. It is split into the chain complex

$$0\leftarrow\mathcal{S}^{0,n}_\infty[F;Y]\overset{\overline{\delta}}{\longleftarrow}\mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]_1\cdots\overset{\overline{\delta}}{\longleftarrow}\mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]_k\cdots \tag{13}$$

graded by the antifield number of its elements. It is readily observed that the boundaries of the complex (13) vanish on-shell.

Note that the homology groups $H_*(\overline{\delta})$ of the complex (13) are $\mathcal{S}^0_\infty[F;Y]$-modules, but these modules fail to be torsion-free. Indeed, given a cycle $\phi\in\mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]_k$ and an element $f=\overline{\delta}\sigma$ of the ring $\mathcal{S}^0_\infty[F;Y]\subset\mathcal{P}^0_\infty[\overline{Y}^*;F;Y;\overline{F}^*]$, we obtain that $f\phi=\overline{\delta}(\sigma\phi)$ is a boundary. Therefore, one can not apply the Künneth formula to the homology of this complex, though any its term $\mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]_k$ is isomorphic to the graded commutative k-tensor product of the $\mathcal{S}^0_\infty[F;Y]$-module $\mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]_1$.

The homology $H_0(\overline{\delta})$ of the complex (13) is not trivial, but this homology and the higher ones $H_{k\geq2}(\overline{\delta})$ are not essential for our consideration. Therefore, we replace the complex (13) with the finite one

$$0\leftarrow\operatorname{Im}\overline{\delta}\overset{\overline{\delta}}{\longleftarrow}\mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]_1\overset{\overline{\delta}}{\longleftarrow}\mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]_2 \tag{14}$$

of graded densities of antifield number $k\leq2$. It is exact at $\operatorname{Im}\overline{\delta}$, and its first homology coincides with that of the complex (13). Let us consider this homology.

A generic one-chain of the complex (14) takes the form

$$\Phi=\sum_{0\leq|\Lambda|}\Phi^{A,\Lambda}\overline{s}_{\Lambda A}\omega, \qquad \Phi^{A,\Lambda}\in\mathcal{S}^0_\infty[F;Y], \tag{15}$$

and the cycle condition $\overline{\delta}\Phi = 0$ reads

$$\sum_{0\leq|\Lambda|} \Phi^{A,\Lambda} d_\Lambda \mathcal{E}_A \omega = 0. \tag{16}$$

One can think of this equality as being a reduction condition on the graded variational derivatives $\mathcal{E}_A$. Conversely, any reduction condition of form (16) comes from some cycle (15). The reduction condition (16) is trivial if a cycle is a boundary, i.e., it takes the form

$$\Phi = \sum_{0\leq|\Lambda|,|\Sigma|} T^{(A\Lambda)(B\Sigma)} d_\Sigma \mathcal{E}_B \overline{s}_{\Lambda A}\omega, \qquad T^{(A\Lambda)(B\Sigma)} = -(-1)^{[A][B]} T^{(B\Sigma)(A\Lambda)}. \tag{17}$$

Definition 5: A graded Lagrangian system is called degenerate if there exist non-trivial reduction conditions (16), called Noether identities.

One can say something more if the $\mathcal{S}^0_\infty[F;Y]$-module $H_1(\overline{\delta})$ is finitely generated, i.e., it possesses the following particular structure. There are elements $\Delta \in H_1(\overline{\delta})$ making up a $\mathbb{Z}_2$-graded projective $C^\infty(X)$-module $\mathcal{C}_{(0)}$ of finite rank which, by virtue of the Serre–Swan theorem, is isomorphic to the module of sections of the product $\overline{V}^* \underset{X}{\times} \overline{E}^*$ of the density-duals of some vector bundles $V\to X$ and $E\to X$. Let $\{\Delta_r\}$ be local bases for this $C^\infty(X)$-module. Every element $\Phi\in H_1(\overline{\delta})$ factorizes

$$\Phi = \sum_{0\leq|\Xi|} G^{r,\Xi} d_\Xi \Delta_r \omega, \qquad G^{r,\Xi}\in \mathcal{S}^0_\infty[F;Y], \tag{18}$$

$$\Delta_r = \sum_{0\leq|\Lambda|} \Delta_r^{A,\Lambda} \overline{s}_{\Lambda A}, \qquad \Delta_r^{A,\Lambda}\in \mathcal{S}^0_\infty[F;Y], \tag{19}$$

via elements of $\mathcal{C}_{(0)}$, i.e., any Noether identity (16) is a corollary of Noether identities

$$\sum_{0\leq|\Lambda|} \Delta_r^{A,\Lambda} d_\Lambda \mathcal{E}_A = 0. \tag{20}$$

Clearly, the factorization (18) is independent of specification of local bases $\{\Delta_r\}$. We say that the Noether identities (20) are complete, and call $\Delta\in\mathcal{C}_{(0)}$ the Noether operators. Note that, being representatives of $H_1(\overline{\delta})$, the graded densities Δ_r (19) are not $\overline{\delta}$-exact.

Proposition 6: If the homology $H_1(\overline{\delta})$ of the complex (14) is finitely generated, this complex can be extended to a one-exact complex with a boundary operator whose nilpotency conditions are just complete Noether identities (see the complex (22) below).

Proof: Let us extend the BGDA $\mathcal{P}^*_\infty[\overline{Y}^*;F;Y;\overline{F}^*]$ to the BGDA $\mathcal{P}^*_\infty[\overline{E}^*\overline{Y}^*;F;Y;\overline{F}^*\overline{V}^*]$ possessing the local basis $\{s^A, \overline{s}_A, \overline{c}_r\}$, where $[\overline{c}_r] = ([\Delta_r]+1)\mathrm{mod}\,2$ and $\mathrm{Ant}[\overline{c}] = 2$. It is provided with the nilpotent graded derivation

$$\delta_0 = \overline{\delta} + \overleftarrow{\partial}^r \Delta_r, \tag{21}$$

called the extended Koszul–Tate differential. Its nilpotency conditions (12) are equivalent to the complete Noether identities (20). Then the module $\mathcal{P}^{0,n}_\infty[\overline{E}^*\overline{Y}^*;F;Y;\overline{F}^*\overline{V}^*]_{\leq 3}$ of graded densities of antifield number $\mathrm{Ant}[\phi]\leq 3$ is split into the chain complex

$$\begin{aligned} 0 \leftarrow \mathrm{Im}\,\overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]_1 \xleftarrow{\delta_0} \mathcal{P}^{0,n}_\infty[\overline{E}^*\overline{Y}^*;F;Y;\overline{F}^*\overline{V}^*]_2 \\ \xleftarrow{\delta_0} \mathcal{P}^{0,n}_\infty[\overline{E}^*\overline{Y}^*;F;Y;\overline{F}^*\overline{V}^*]_3. \end{aligned} \tag{22}$$

Let $H_*(\delta_0)$ denote its homology. We have $H_0(\delta_0)=H_0(\overline{\delta})=0$. Furthermore, any one-cycle Φ up to a boundary takes the form (18) and, therefore, it is a δ_0-boundary

$$\Phi=\sum_{0\leq|\Sigma|} G^{r,\Xi}d_\Xi\Delta_r\omega=\delta_0(\sum_{0\leq|\Sigma|} G^{r,\Xi}\overline{c}_{\Xi r}\omega).$$

Hence, $H_1(\delta_0)=0$, i.e., the complex (22) is one-exact.

IV. THE KOSZUL-TATE COMPLEX OF NOETHER IDENTITIES

Turn now to the homology $H_2(\delta_0)$ of the complex (22). A generic two-chain reads

$$\begin{aligned} &\Phi=G+H=\sum_{0\leq|\Lambda|} G^{r,\Lambda}\overline{c}_{\Lambda r}\omega+\sum_{0\leq|\Lambda|,|\Sigma|} H^{(A,\Lambda)(B,\Sigma)}\overline{s}_{\Lambda A}\overline{s}_{\Sigma B}\omega, \\ &G^{r,\Lambda}\in\mathcal{S}^0_\infty[F;Y], \qquad H^{(A,\Lambda)(B,\Sigma)}\mathcal{V}\in\mathcal{S}^0_\infty[F;Y], \qquad \mathcal{V}\in\mathcal{O}^nX. \end{aligned} \tag{23}$$

The cycle condition $\delta_0\Phi=0$ takes the form

$$\sum_{0\leq|\Lambda|} G^{r,\Lambda}d_\Lambda\Delta_r\omega+\overline{\delta}H=0. \tag{24}$$

One can think of this equality as being the reduction condition on the Noether operators (19). Conversely, let

$$\Phi=\sum_{0\leq|\Lambda|} G^{r,\Lambda}\overline{c}_{\Lambda r}\omega\in\mathcal{P}^{0,n}_\infty[\overline{E}^*\overline{Y}^*;F;Y;\overline{F}^*\overline{V}^*]_2$$

be a graded density such that the reduction condition (24) holds. Obviously, it is a cycle condition of the two-chain (23). The reduction condition (24) is trivial either if a two-cycle Φ (23) is a boundary or its summand G vanishes on-shell.

Definition 7: A degenerate graded Lagrangian system in Proposition 6 is said to be one-stage reducible if there exist non-trivial reduction conditions (24), called the first-stage Noether identities.

Proposition 8: First-stage Noether identities can be identified to nontrivial elements of the homology $H_2(\delta_0)$ iff any $\overline{\delta}$-cycle $\phi\in\mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]_2$ is a δ_0-boundary.

Proof: It suffices to show that, if the summand G of a two-cycle Φ (23) is $\overline{\delta}$-exact, then Φ is a boundary. If $G = \overline{\delta}\Psi$, then

$$\Phi = \delta_0\Psi + (\overline{\delta} - \delta_0)\Psi + H. \tag{25}$$

The cycle condition reads

$$\delta_0\Phi = \overline{\delta}((\overline{\delta} - \delta_0)\Psi + H) = 0.$$

Then $(\overline{\delta} - \delta_0)\Psi + H$ is δ_0-exact since any $\overline{\delta}$-cycle $\phi \in \mathcal{P}^{0,n}_\infty[\overline{Y}^*; F; Y; \overline{F}^*]_2$, by assumption, is a δ_0-boundary. Consequently, Φ (25) is δ_0-exact. Conversely, let $\Phi \in \mathcal{P}^{0,n}_\infty[\overline{Y}^*; F; Y; \overline{F}^*]_2$ be an arbitrary $\overline{\delta}$-cycle. The cycle condition reads

$$\overline{\delta}\Phi = 2\Phi^{(A,\Lambda)(B,\Sigma)}\overline{s}_{\Lambda A}\overline{\delta}\overline{s}_{\Sigma B}\omega = 2\Phi^{(A,\Lambda)(B,\Sigma)}\overline{s}_{\Lambda A}d_\Sigma\mathcal{E}_B\omega = 0. \tag{26}$$

It follows that $\Phi^{(A,\Lambda)(B,\Sigma)}\overline{\delta}\overline{s}_{\Sigma B} = 0$ for all indices (A,Λ). Omitting a $\overline{\delta}$-boundary term, we obtain

$$\Phi^{(A,\Lambda)(B,\Sigma)}\overline{s}_{\Sigma B} = G^{(A,\Lambda)(r,\Xi)}d_\Xi\Delta_r.$$

Hence, Φ takes the form

$$\Phi = G'^{(A,\Lambda)(r,\Xi)}d_\Xi\Delta_r\overline{s}_{\Lambda A}\omega. \tag{27}$$

We can associate to it the three-chain

$$\Psi = G'^{(A,\Lambda)(r,\Xi)}\overline{c}_{\Xi r}\overline{s}_{\Lambda A}\omega$$

such that

$$\delta_0\Psi = \Phi + \sigma = \Phi + G''^{(A,\Lambda)(r,\Xi)}d_\Lambda\mathcal{E}_A\overline{c}_{\Xi r}\omega.$$

Owing to the equality $\overline{\delta}\Phi = 0$, we have $\delta_0\sigma = 0$. Since σ is $\overline{\delta}$-exact, it by assumption is δ_0-exact, i.e., $\sigma = \delta_0\psi$. Then we obtain that $\Phi = \delta_0\Psi - \delta_0\psi$.

Lemma 9: It is easily justified that a two-cycle $\Phi \in \mathcal{P}^{0,n}_\infty[\overline{Y}^*; F; Y; \overline{F}^*]_2$ is δ_0-exact iff Φ up to a $\overline{\delta}$-boundary takes the form

$$\Phi = \sum_{0\leq|\Lambda|,|\Sigma|} G'^{(r,\Sigma)(r',\Lambda)}d_\Sigma\Delta_r d_\Lambda\Delta_{r'}\omega. \tag{28}$$

If the condition of Proposition 8 (called the two-homology regularity condition) is satisfied, let us assume that the first-stage Noether identities are finitely generated as follows. There are elements $\Delta_{(1)} \in H_2(\delta_0)$ making up a $\mathbb{Z}_2$-graded projective $C^\infty(X)$-module $\mathcal{C}_{(1)}$ of finite rank which is isomorphic to the module of sections of the product $\overline{V}^*_1 \underset{X}{\times} \overline{E}^*_1$ of the

density-duals of some vector bundles $V_1 \to X$ and $E_1 \to X$. Let $\{\Delta_{r_1}\}$ be local bases for this $C^\infty(X)$-module. Every element $\Phi \in H_2(\delta_0)$ factorizes

$$\Phi = \sum_{0\leq|\Xi|} \Phi^{r_1,\Xi} d_\Xi \Delta_{r_1}\omega, \qquad \Phi^{r_1,\Xi} \in \mathcal{S}^0_\infty[F;Y], \tag{29}$$

$$\Delta_{r_1} = G_{r_1} + h_{r_1} = \sum_{0\leq|\Lambda|} \Delta^{r,\Lambda}_{r_1}\overline{c}_{\Lambda r} + h_{r_1}, \qquad h_{r_1}\omega \in \mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*], \tag{30}$$

via elements of $\mathcal{C}_{(1)}$, i.e., any first-stage Noether identity (24) results from the equalities

$$\sum_{0\leq|\Lambda|} \Delta^{r,\Lambda}_{r_1} d_\Lambda \Delta_r + \overline{\delta} h_{r_1} = 0, \tag{31}$$

called the complete first-stage Noether identities. Elements of $\mathcal{C}_{(1)}$ are called the first-stage Noether operators. Note that first summands G_{r_1} of operators Δ_{r_1} (30) are not $\overline{\delta}$-exact.

Proposition 10: Given a reducible degenerate Lagrangian system, let the associated one-exact complex (22) obey the two-homology regularity condition and let its homology $H_2(\delta_0)$ (first-stage Noether identities) be finitely generated. Then this complex is extended to the two-exact one with a boundary operator whose nilpotency conditions are equivalent to complete Noether and first-stage Noether identities (see the complex (33) below).

Proof: Let us consider the BGDA $\mathcal{P}^*_\infty[\overline{E}_1^*\overline{E}^*\overline{Y}^*;F;Y;\overline{F}^*\overline{V}^*\overline{V}_1^*]$ with the local basis $\{s^A,\overline{s}_A,\overline{c}_r,\overline{c}_{r_1}\}$, where $[\overline{c}_{r_1}] = ([\Delta_{r_1}]+1)\mathrm{mod}\,2$ and $\mathrm{Ant}[\overline{c}_{r_1}] = 3$. It can be provided the first-stage Koszul–Tate differential defined as the nilpotent graded derivation

$$\delta_1 = \delta_0 + \overleftarrow{\partial}^{r_1}\Delta_{r_1}. \tag{32}$$

Its nilpotency conditions (12) are equivalent to complete Noether identities (20) and complete first-stage Noether identities (31). Then the module $\mathcal{P}^{0,n}_\infty[\overline{E}_1^*\overline{E}^*\overline{Y}^*;F;Y;\overline{F}^*\overline{V}^*\overline{V}_1^*]_{\leq 4}$ of graded densities of antifield number $\mathrm{Ant}[\phi]\leq 4$ is split into the chain complex

$$\begin{array}{r}0 \leftarrow \mathrm{Im}\,\overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]_1 \xleftarrow{\delta_0} \mathcal{P}^{0,n}_\infty[\overline{E}^*\overline{Y}^*;F;Y;\overline{F}^*\overline{V}^*]_2 \xleftarrow{\delta_1} \\ \mathcal{P}^{0,n}_\infty[\overline{E}_1^*\overline{E}^*\overline{Y}^*;F;Y;\overline{F}^*\overline{V}^*\overline{V}_1^*]_3 \xleftarrow{\delta_1} \mathcal{P}^{0,n}_\infty[\overline{E}_1^*\overline{E}^*\overline{Y}^*;F;Y;\overline{F}^*\overline{V}^*\overline{V}_1^*]_4.\end{array} \tag{33}$$

Let $H_*(\delta_1)$ denote its homology. It is readily observed that

$$H_0(\delta_1) = H_0(\overline{\delta}), \qquad H_1(\delta_1) = H_1(\delta_0) = 0.$$

By virtue of the expression (29), any two-cycle of the complex (33) is a boundary

$$\Phi = \sum_{0\leq|\Xi|} \Phi^{r_1,\Xi} d_\Xi \Delta_{r_1}\omega = \delta_1\Big(\sum_{0\leq|\Xi|} \Phi^{r_1,\Xi}\overline{c}_{\Xi r_1}\Big)\omega.$$

It follows that $H_2(\delta_1) = 0$, i.e., the complex (33) is two-exact.

If the third homology $H_3(\delta_1)$ of the complex (33) is not trivial, there are reduction conditions on the first-stage Noether operators, and so on. Iterating the arguments, we come to the following.

Let $(\mathcal{S}^*_\infty[F;Y],L)$ be a degenerate graded Lagrangian system whose Noether identities are finitely generated. In accordance with Proposition 6, we associates to it the one-exact chain complex (22). Given an integer $N\geq 1$, let $V_1,\ldots,V_N,E_1,\ldots,E_N$ be some vector bundles over X and

$$\mathcal{P}^*_\infty\{N\}=\mathcal{P}^*_\infty[\overline{E}^*_N\cdots\overline{E}^*_1\overline{E}^*\overline{Y}^*;F;Y;\overline{F}^*\overline{V}^*\overline{V}^*_1\cdots\overline{V}^*_N] \tag{34}$$

a BGDA with local bases $\{s^A,\overline{s}_A,\overline{c}_r,\overline{c}_{r_1},\ldots,\overline{c}_{r_N}\}$ graded by antifield numbers $\mathrm{Ant}[\overline{c}_{r_k}]=k+2$. Let $k=-1,0$ further stand for $\overline{s}_A$ and $\overline{c}_r$, respectively. We assume that:

(i) the BGDA $\mathcal{P}^*_\infty\{N\}$ (34) is provided with a nilpotent graded derivation

$$\delta_N=\delta_0+\sum_{1\leq k\leq N}\overleftarrow{\partial}^{r_k}\Delta_{r_k}, \tag{35}$$

$$\Delta_{r_k}=G_{r_k}+h_{r_k}=\sum_{0\leq|\Lambda|}\Delta^{r_{k-1},\Lambda}_{r_k}\overline{c}_{\Lambda r_{k-1}}+\sum_{0\leq\Sigma,0\leq\Xi}(h^{(A,\Xi)(r_{k-2},\Sigma)}_{r_k}\overline{s}_{\Xi A}\overline{c}_{\Sigma r_{k-2}}+\ldots), \tag{36}$$

of antifield number -1;

(ii) the module $\mathcal{P}^{0,n}_\infty\{N\}_{\leq N+3}$ of graded densities of antifield number $\mathrm{Ant}[\phi]\leq N+3$ is split into the $(N+1)$-exact chain complex

$$\begin{aligned}0\leftarrow \mathrm{Im}\,\overline{\delta}\xleftarrow{\overline{\delta}}\mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]_1\xleftarrow{\delta_0}\mathcal{P}^{0,n}_\infty\{0\}_2\xleftarrow{\delta_1}\mathcal{P}^{0,n}_\infty\{1\}_3\cdots\\ \xleftarrow{\delta_{N-1}}\mathcal{P}^{0,n}_\infty\{N-1\}_{N+1}\xleftarrow{\delta_N}\mathcal{P}^{0,n}_\infty\{N\}_{N+2}\xleftarrow{\delta_N}\mathcal{P}^{0,n}_\infty\{N\}_{N+3},\end{aligned} \tag{37}$$

which satisfies the $(N+1)$-homology regularity condition in accordance with forthcoming Definition 11.

Definition 11: One says that the complex (37) obeys the $(N+1)$-homology regularity condition if any $\delta_{k<N-1}$-cycle $\phi\in\mathcal{P}^{0,n}_\infty\{k\}_{k+3}\subset\mathcal{P}^{0,n}_\infty\{k+1\}_{k+3}$ is a δ_{k+1}-boundary.

Remark 3: The $(N+1)$-exactness of the complex (37) implies that any $\delta_{k<N-1}$-cycle $\phi\in\mathcal{P}^{0,n}_\infty\{k\}_{k+3}$, $k<N$, is a δ_{k+2}-boundary, but not necessary a δ_{k+1}-one.

If $N=1$, the complex $\mathcal{P}^{0,n}_\infty\{1\}_{\leq 4}$ (37) restarts the complex (33) associated to a first-stage reducible graded Lagrangian system by virtue of Proposition 10. Therefore, we agree to call δ_N (35) the N-stage Koszul–Tate differential. Its nilpotency implies complete Noether identities (20), first-stage Noether identities (31) and the equalities

$$\sum_{0\leq|\Lambda|}\Delta^{r_{k-1},\Lambda}_{r_k}d_\Lambda(\sum_{0\leq|\Sigma|}\Delta^{r_{k-2},\Sigma}_{r_{k-1}}\overline{c}_{\Sigma r_{k-2}})+\overline{\delta}(\sum_{0\leq\Sigma,0\leq\Xi}h^{(A,\Xi)(r_{k-2},\Sigma)}_{r_k}\overline{s}_{\Xi A}\overline{c}_{\Sigma r_{k-2}})=0, \tag{38}$$

for $k=2,\ldots,N$. One can think of the equalities (38) as being complete k-stage Noether identities because of their properties which we will justify in the case of $k=N+1$. Accordingly, Δ_{r_k} (36) are said to be the k-stage Noether operators.

Let us consider the $(N+2)$-homology of the complex (37). A generic $(N+2)$-chain $\Phi\in\mathcal{P}^{0,n}_\infty\{N\}_{N+2}$ takes the form

$$\Phi=G+H=\sum_{0\leq|\Lambda|}G^{r_N,\Lambda}\overline{c}_{\Lambda r_N}\omega+\sum_{0\leq\Sigma,0\leq\Xi}(H^{(A,\Xi)(r_{N-1},\Sigma)}\overline{s}_{\Xi A}\overline{c}_{\Sigma r_{N-1}}+...)\omega. \qquad (39)$$

Let it be a cycle. The cycle condition $\delta_N\Phi=0$ implies the equality

$$\sum_{0\leq|\Lambda|}G^{r_N,\Lambda}d_\Lambda(\sum_{0\leq|\Sigma|}\Delta^{r_{N-1},\Sigma}_{r_N}\overline{c}_{\Sigma r_{N-1}})+\overline{\delta}(\sum_{0\leq\Sigma,0\leq\Xi}H^{(A,\Xi)(r_{N-1},\Sigma)}\overline{s}_{\Xi A}\overline{c}_{\Sigma r_{N-1}})=0. \qquad (40)$$

One can think of this equality as being the reduction condition on the N-stage Noether operators (36). Conversely, let

$$\Phi=\sum_{0\leq|\Lambda|}G^{r_N,\Lambda}\overline{c}_{\Lambda r_N}\omega\in\mathcal{P}^{0,n}_\infty\{N\}_{N+2}$$

be a graded density such that the reduction condition (40) holds. Then this reduction condition can be extended to a cycle one as follows. It is brought into the form

$$\begin{aligned}\delta_N(\sum_{0\leq|\Lambda|}G^{r_N,\Lambda}\overline{c}_{\Lambda r_N}+\sum_{0\leq\Sigma,0\leq\Xi}H^{(A,\Xi)(r_{N-1},\Sigma)}\overline{s}_{\Xi A}\overline{c}_{\Sigma r_{N-1}})=\\ -\sum_{0\leq|\Lambda|}G^{r_N,\Lambda}d_\Lambda h_{r_N}+\sum_{0\leq\Sigma,0\leq\Xi}H^{(A,\Xi)(r_{N-1},\Sigma)}\overline{s}_{\Xi A}d_\Sigma\Delta_{r_{N-1}}.\end{aligned}$$

A glance at the expression (36) shows that the term in the right-hand side of this equality belongs to $\mathcal{P}^{0,n}_\infty\{N-2\}_{N+1}$. It is a δ_{N-2}-cycle and, consequently, a δ_{N-1}-boundary $\delta_{N-1}\Psi$ in accordance with the $(N+1)$-homology regularity condition. Then the reduction condition (40) is a $\overline{c}_{\Sigma r_{N-1}}$-dependent part of the cycle condition

$$\delta_N(\sum_{0\leq|\Lambda|}G^{r_N,\Lambda}\overline{c}_{\Lambda r_N}+\sum_{0\leq\Sigma,0\leq\Xi}H^{(A,\Xi)(r_{N-1},\Sigma)}\overline{s}_{\Xi A}\overline{c}_{\Sigma r_{N-1}}-\Psi)=0,$$

but $\delta_N\Psi$ does not make a contribution to this reduction condition.

Being a cycle condition, the reduction condition (40) is trivial either if a cycle Φ (39) is a δ_N-boundary or its summand G is $\overline{\delta}$-exact, i.e., it is a boundary, too, as we have stated above. Then Definition 7 can be generalized as follows.

Definition 12: A degenerate graded Lagrangian system is said to be $(N+1)$-stage reducible if there exist non-trivial reduction conditions (40), called the $(N+1)$-stage Noether identities.

Theorem 13: (i) The $(N+1)$-stage Noether identities can be identified to nontrivial elements of the homology $H_{N+2}(\delta_N)$ of the complex (37) iff this homology obeys the $(N+2)$-homology regularity condition. (ii) If the homology $H_{N+2}(\delta_N)$ is finitely generated as defined below, the complex (37) admits an $(N+2)$-exact extension.

Proof: (i) The $(N+2)$-homology regularity condition implies that any δ_{N-1}-cycle $\Phi \in \mathcal{P}^{0,n}_\infty\{N-1\}_{N+2} \subset \mathcal{P}^{0,n}_\infty\{N\}_{N+2}$ is a δ_N-boundary. Therefore, if Φ (39) is a representative of a nontrivial element of $H_{N+2}(\delta_N)$, its summand G linear in $\overline{c}_{\Lambda r_N}$ does not vanish. Moreover, it is not a $\overline{\delta}$-boundary. Indeed, if $\Phi = \overline{\delta}\Psi$, then

$$\Phi = \delta_N\Psi + (\overline{\delta} - \delta_N)\Psi + H. \tag{41}$$

The cycle condition takes the form

$$\delta_N\Phi = \delta_{N-1}((\overline{\delta} - \delta_N)\Psi + H) = 0.$$

Hence, $(\overline{\delta}-\delta_N)\Psi+H$ is δ_N-exact since any δ_{N-1}-cycle $\phi \in \mathcal{P}^{0,n}_\infty\{N-1\}_{N+2}$ is a δ_N-boundary. Consequently, Φ (41) is a boundary. If the $(N+2)$-homology regularity condition does not hold, trivial reduction conditions (40) also come from nontrivial elements of the homology $H_{N+2}(\delta_N)$. (ii) Let the $(N+1)$-stage Noether identities be finitely generated. Namely, there exist elements $\Delta_{(N+1)} \in H_{N+2}(\delta_N)$ making up a $\mathbb{Z}_2$-graded projective $C^\infty(X)$-module $\mathcal{C}_{(N+1)}$ of finite rank which is isomorphic to the module of sections of the product $\overline{V}^*_{N+1} \underset{X}{\times} \overline{E}^*_{N+1}$ of the density-duals of some vector bundles $V_{N+1} \to X$ and $E_{N+1} \to X$. Let $\{\Delta_{r_{N+1}}\}$ be local bases for this $C^\infty(X)$-module. Then any element $\Phi \in H_{N+2}(\delta_N)$ factorizes

$$\Phi = \sum_{0\leq|\Xi|} \Phi^{r_{N+1},\Xi} d_\Xi \Delta_{r_{N+1}}\omega, \qquad \Phi^{r_{N+1},\Xi} \in \mathcal{S}^0_\infty[F;Y], \tag{42}$$

$$\Delta_{r_{N+1}} = G_{r_{N+1}} + h_{r_{N+1}} = \sum_{0\leq|\Lambda|} \Delta^{r_N,\Lambda}_{r_{N+1}} \overline{c}_{\Lambda r_N} + h_{r_{N+1}}, \tag{43}$$

via elements of $\mathcal{C}_{(N+1)}$. Clearly, this factorization is independent of specification of local bases $\{\Delta_{r_{N+1}}\}$. Let us extend the BGDA $\mathcal{P}^*_\infty\{N\}$ (34) to the BGDA $\mathcal{P}^*_\infty\{N+1\}$ possessing local bases

$$\{s^A, \overline{s}_A, \overline{c}_r, \overline{c}_{r_1}, \ldots, \overline{c}_{r_N}, \overline{c}_{r_{N+1}}\}, \quad \mathrm{Ant}[\overline{c}_{r_{N+1}}] = N+3, \quad [\overline{c}_{r_{N+1}}] = ([\Delta_{r_{N+1}}]+1) \bmod 2.$$

It is provided with the nilpotent graded derivation

$$\delta_{N+1} = \delta_N + \overleftarrow{\partial}^{r_{N+1}} \Delta_{r_{N+1}} \tag{44}$$

of antifield number -1. With this graded derivation, the module $\mathcal{P}^{0,n}_\infty\{N+1\}_{\leq N+4}$ of graded densities of antifield number $\mathrm{Ant}[\phi] \leq N+4$ is split into the chain complex

$$\begin{aligned} 0 \leftarrow \mathrm{Im}\,\overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{Y}^*;F;Y;\overline{F}^*]_1 \xleftarrow{\delta_0} \mathcal{P}^{0,n}_\infty\{0\}_2 \xleftarrow{\delta_1} \mathcal{P}^{0,n}_\infty\{1\}_3 \cdots \\ \xleftarrow{\delta_{N-1}} \mathcal{P}^{0,n}_\infty\{N-1\}_{N+1} \xleftarrow{\delta_N} \mathcal{P}^{0,n}_\infty\{N\}_{N+2} \xleftarrow{\delta_{N+1}} \mathcal{P}^{0,n}_\infty\{N+1\}_{N+3} \xleftarrow{\delta_{N+1}} \mathcal{P}^{0,n}_\infty\{N+1\}_{N+4}. \end{aligned} \tag{45}$$

It is readily observed that this complex is $(N+2)$-exact. In this case, the $(N+1)$-stage Noether identities (40) come from the complete $(N+1)$-stage Noether identities

$$\sum_{0\leq|\Lambda|}\Delta^{r_N,\Lambda}_{r_{N+1}}d_\Lambda\Delta_r\omega+\overline{\delta}h_{r_{N+1}}\omega=0,$$

which are reproduced as the nilpotency conditions of the graded derivation (44).

The iteration procedure based on Theorem 13 can be prolonged up to an integer $N_{\max}$ when the $N_{\max}$-stage Noether identities are irreducible, i.e., the homology $H_{N_{\max}+2}(\delta_{N_{\max}})$ is trivial. This iteration procedure may also be infinite. It results in the manifested exact Koszul–Tate complex with the Koszul–Tate boundary operator whose nilpotency conditions reproduce all Noether and higher Noether identities of an original Lagrangian system.

V. EXAMPLE

Let us consider a fiber bundle

$$Y=\mathbb{R}\underset{X}{\times}\overset{n-1}{\wedge}T^*X, \tag{46}$$

coordinated by $(x^\lambda,A,B_{\mu_1\ldots\mu_{n-1}})$. The corresponding BGDA is $\mathcal{S}^*_\infty[Y]=\mathcal{O}^*_\infty Y$. There is the canonical $(n-1)$-form

$$B=\frac{1}{(n+1)!}B_{\mu_1\ldots\mu_{n-1}}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{n-1}}\in\mathcal{O}^*_\infty Y$$

on Y (46). A Lagrangian of topological BF theory in question reads

$$L_{\rm BF}=\frac{1}{n}Ad_HB. \tag{47}$$

The corresponding Euler–Lagrange operator (9) takes the form

$$\begin{aligned}&\delta L=dA\wedge\mathcal{E}\omega+dB_{\mu_1\ldots\mu_{n-1}}\wedge\mathcal{E}^{\mu_1\ldots\mu_{n-1}}\omega\\&\mathcal{E}=\epsilon^{\mu\mu_1\ldots\mu_{n-1}}d_\mu B_{\mu_1\ldots\mu_{n-1}},\qquad \mathcal{E}^{\mu_1\ldots\mu_{n-1}}=-\epsilon^{\mu\mu_1\ldots\mu_{n-1}}d_\mu A,\end{aligned} \tag{48}$$

where ϵ is the Levi–Civita symbol.

Let us extend the BGDA $\mathcal{O}^*_\infty Y$ to the BGDA $\mathcal{P}^*_\infty[\overline{Y}^*;Y]$ where

$$VY=Y\underset{X}{\times}Y,\qquad \overline{Y}^*=(\mathbb{R}\underset{X}{\times}\overset{n-1}{\wedge}TX)\underset{X}{\otimes}\overset{n}{\wedge}T^*X.$$

This BGDA possesses the local bases $\{A,B_{\mu_1\ldots\mu_{n-1}},\overline{s},\overline{s}^{\mu_1\ldots\mu_{n-1}}\}$, where $\overline{s},\overline{s}^{\mu_1\ldots\mu_{n-1}}$ are odd of antifield number 1. With the nilpotent Koszul–Tate differential

$$\overline{\delta}=\frac{\overleftarrow{\partial}}{\partial\overline{s}}\mathcal{E}+\frac{\overleftarrow{\partial}}{\partial\overline{s}^{\mu_1\ldots\mu_{n-1}}}\mathcal{E}^{\mu_1\ldots\mu_{n-1}},$$

we have the complex (14):

$$0 \leftarrow \mathrm{Im}\,\overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{Y}^*;Y]_1 \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{Y}^*;Y]_2.$$

A generic one-chain reads

$$\Phi = \sum_{0\leq|\Lambda|} (\Phi^\Lambda \overline{s}_\Lambda + \Phi^\Lambda_{\mu_1\ldots\mu_{n-1}} \overline{s}^{\mu_1\ldots\mu_{n-1}}_\Lambda)\omega,$$

and the cycle condition $\overline{\delta}\Phi = 0$ takes the form

$$\Phi^\Lambda \mathcal{E}_\Lambda + \Phi^\Lambda_{\mu_1\ldots\mu_{n-1}} \mathcal{E}^{\mu_1\ldots\mu_{n-1}}_\Lambda = 0. \tag{49}$$

If Φ^Λ and $\Phi^\Lambda_{\mu_1\ldots\mu_{n-1}}$ are independent of the variational derivatives (48) (i.e., Φ is a nontrivial cycle), the equality (49) is split into the following two ones

$$\Phi^\Lambda \mathcal{E}_\Lambda = 0, \tag{50}$$

$$\Phi^\Lambda_{\mu_1\ldots\mu_{n-1}} \mathcal{E}^{\mu_1\ldots\mu_{n-1}}_\Lambda = 0. \tag{51}$$

The equality (50) holds iff $\Phi^\Lambda = 0$, i.e., there is no Noether identities for $\mathcal{E}$. The equality (51) is satisfied iff

$$\Phi^{\lambda_1\ldots\lambda_k}_{\mu_1\ldots\mu_{n-1}} \epsilon^{\mu\mu_1\ldots\mu_{n-1}} = -\Phi^{\mu\lambda_2\ldots\lambda_k}_{\mu_1\ldots\mu_{n-1}} \epsilon^{\lambda_1\mu_1\ldots\mu_{n-1}}.$$

It follows that Φ factorizes as

$$\Phi = \sum_{0\leq|\Xi|} G^\Xi_{\nu_2\ldots\nu_{n-1}} d_\Xi \Delta^{\nu_2\ldots\nu_{n-1}}\omega$$

via local graded densities

$$\Delta^{\nu_2\ldots\nu_{n-1}} = \Delta^{\nu_2\ldots\nu_{n-1},\lambda}_{\alpha_1\ldots\alpha_{n-1}} \overline{s}^{\alpha_1\ldots\alpha_{n-1}}_\lambda = \delta^\lambda_{\alpha_1}\delta^{\nu_2}_{\alpha_2}\cdots\delta^{\nu_{n-1}}_{\alpha_{n-1}} \overline{s}^{\alpha_1\ldots\alpha_{n-1}}_\lambda = d_{\nu_1}\overline{s}^{\nu_1\nu_2\ldots\nu_{n-1}}, \tag{52}$$

which provide the complete Noether identities[1]

$$d_{\nu_1}\mathcal{E}^{\nu_1\nu_2\ldots\nu_{n-1}} = 0. \tag{53}$$

The local graded densities (52) form the bases of a projective $C^\infty(X)$-module of finite rank which is isomorphic to the module of sections of the vector bundle

$$\overline{V}^* = \overset{n-2}{\wedge} TX \underset{X}{\otimes} \overset{n}{\wedge} T^*X, \qquad V = \overset{n-2}{\wedge} T^*X.$$

Therefore, let us extend the BGDA $\mathcal{P}^*_\infty[\overline{Y}^*;Y]$ to the BGDA $\mathcal{P}^*_\infty\{0\} = \mathcal{P}^*_\infty[\overline{Y}^*;Y;V]$ possessing the local bases

$$\{A, B_{\mu_1\ldots\mu_{n-1}}, \overline{s}, \overline{s}^{\mu_1\ldots\mu_{n-1}}, \overline{c}^{\mu_2\ldots\mu_{n-1}}\},$$

where $\overline{c}^{\mu_2\cdots\mu_{n-1}}$ are even of antifield number 2. Let

$$\delta_0=\overline{\delta}+\frac{\overleftarrow{\partial}}{\partial\overline{c}^{\mu_2\cdots\mu_{n-1}}}\Delta^{\mu_2\cdots\mu_{n-1}}$$

be its nilpotent graded derivation. Its nilpotency is equivalent to the Noether identities (53). Then have the one-exact complex

$$0\leftarrow \mathrm{Im}\,\overline{\delta}\xleftarrow{\overline{\delta}}\mathcal{P}^{0,n}_\infty[\overline{Y}^*;Y]_1\xleftarrow{\delta_0}\mathcal{P}^{0,n}_\infty\{0\}_2\xleftarrow{\delta_0}\mathcal{P}^{0,n}_\infty\{0\}_3,$$

and so on. Iterating the arguments we come to the following $(N+1)$-exact complex (37) for $N\leq n-3$.

Let us consider the vector bundles

$$V_k=\overset{n-k-2}{\wedge}T^*X,\qquad k=1,\ldots,N,$$

and the corresponding BGDA

$$\mathcal{P}^*_\infty\{N\}=\mathcal{P}^*_\infty[...V_3V_1\overline{Y}^*;Y;VV_2V_4...],$$

possessing the local bases

$$\begin{array}{l}\{A,B_{\mu_1\ldots\mu_{n-1}},\overline{s},\overline{s}^{\mu_1\cdots\mu_{n-1}},\overline{c}^{\mu_2\cdots\mu_{n-1}},\ldots,\overline{c}^{\mu_{N+2}\cdots\mu_{n-1}}\},\\ [\overline{c}^{\mu_{k+2}\cdots\mu_{n-1}}]=(k+1)\,\mathrm{mod}\,2,\qquad \mathrm{Ant}[\overline{c}^{\mu_{k+2}\cdots\mu_{n-1}}]=k+3.\end{array}$$

It is provided with the nilpotent graded derivation

$$\begin{aligned}&\delta_N=\delta_0+\sum_{1\leq k\leq N}\frac{\overleftarrow{\partial}}{\partial\overline{c}^{\mu_{k+2}\cdots\mu_{n-1}}}\Delta^{\mu_{k+2}\cdots\mu_{n-1}},\\ &\Delta^{\mu_{k+2}\cdots\mu_{n-1}}=d_{\mu_{k+1}}c^{\mu_{k+1}\mu_{k+2}\cdots\mu_{n-1}},\end{aligned}\tag{54}$$

of antifield number -1. The nilpotency results from the Noether identities (53) and the equalities

$$d_{\mu_{k+2}}\Delta^{\mu_{k+2}\cdots\mu_{n-1}}=0,\qquad ,k=0,\ldots,N,\tag{55}$$

which are k-stage Noether identities.[1] Then the above mentioned $(N+1)$-exact complex is

$$\begin{aligned}&0\leftarrow \mathrm{Im}\,\overline{\delta}\xleftarrow{\overline{\delta}}\mathcal{P}^{0,n}_\infty[\overline{Y}^*;Y]_1\xleftarrow{\delta_0}\mathcal{P}^{0,n}_\infty\{0\}_2\xleftarrow{\delta_1}\mathcal{P}^{0,n}_\infty\{1\}_3\cdots\\ &\xleftarrow{\delta_{N-1}}\mathcal{P}^{0,n}_\infty\{N-1\}_{N+1}\xleftarrow{\delta_N}\mathcal{P}^{0,n}_\infty\{N\}_{N+2}\xleftarrow{\delta_N}\mathcal{P}^{0,n}_\infty\{N\}_{N+3}.\end{aligned}\tag{56}$$

It obeys the $(N+2)$-homology regularity condition as follows.

Lemma 14: Any $(N+2)$-cycle $\Phi\in\mathcal{P}^{0,n}_\infty\{N-1\}_{N+2}$ up to a δ_{N-1}-boundary takes the form

$$\Phi=\sum_{k_1+\cdots+k_i+3i=N+2}\ \sum_{0\leq|\Lambda_1|,\ldots,|\Lambda_i|} G^{\Lambda_1\cdots\Lambda_i}_{\mu^1_{k_1+2}\cdots\mu^1_{n-1};\ldots;\mu^i_{k_i+2}\cdots\mu^i_{n-1}} \tag{57}$$
$$d_{\Lambda_1}\Delta^{\mu^1_{k_1+2}\cdots\mu^1_{n-1}}\cdots d_{\Lambda_i}\Delta^{\mu^i_{k_i+2}\cdots\mu^i_{n-1}}\omega, \qquad k=-1,0,1,\ldots,N,$$

where $k=-1$ stands for

$$\overline{c}^{\mu_1\cdots\mu_{n-1}}=\overline{s}^{\mu_1\cdots\mu_{n-1}}, \qquad \Delta^{\mu_1\cdots\mu_{n-1}}=\mathcal{E}^{\mu_1\cdots\mu_{n-1}}.$$

It follows that Φ is a δ_N-boundary.

Proof: Let us choose some basis element $\overline{c}^{\mu_{k+2}\cdots\mu_{n-1}}$ and denote it simply by $\overline{c}$. Let Φ contain a summand $\phi_1\overline{c}$, linear in $\overline{c}$. Then the cycle condition reads

$$\delta_{N-1}\Phi=\delta_{N-1}(\Phi-\phi_1\overline{c})+(-1)^{[\overline{c}]}\delta_{N-1}(\phi_1)\overline{c}+\phi\Delta=0, \qquad \Delta=\delta_{N-1}\overline{c}.$$

It follows that Φ contains a summand $\psi\Delta$ such that

$$(-1)^{[\overline{c}]+1}\delta_{N-1}(\psi)\Delta+\phi\Delta=0.$$

This equality implies the relation

$$\phi_1=(-1)^{[\overline{c}]+1}\delta_{N-1}(\psi) \tag{58}$$

because the reduction conditions (55) involve total derivatives of Δ, but not Δ. Hence,

$$\Phi=\Phi'+\delta_{N-1}(\psi\overline{c}),$$

where Φ' contains no term linear in $\overline{c}$. Furthermore, let $\overline{c}$ be even and Φ has a summand $\sum\phi_r\overline{c}^r$ polynomial in $\overline{c}$. Then the cycle condition leads to the equalities

$$\phi_r\Delta=-\delta_{N-1}\phi_{r-1}, \qquad r\geq 2.$$

Since ϕ_1 (58) is δ_{N-1}-exact, then $\phi_2=0$ and, consequently, $\phi_{r>2}=0$. Thus, a cycle Φ up to a δ_{N-1}-boundary contains no term polynomial in $\overline{c}$. It reads

$$\Phi=\sum_{k_1+\cdots+k_i+3i=N+2}\ \sum_{0<|\Lambda_1|,\ldots,|\Lambda_i|} G^{\Lambda_1\cdots\Lambda_i}_{\mu^1_{k_1+2}\cdots\mu^1_{n-1};\ldots;\mu^i_{k_i+2}\cdots\mu^i_{n-1}}\overline{c}^{\mu^1_{k_1+2}\cdots\mu^1_{n-1}}_{\Lambda_1}\cdots\overline{c}^{\mu^i_{k_i+2}\cdots\mu^i_{n-1}}_{\Lambda_i}\omega. \tag{59}$$

However, the terms polynomial in $\overline{c}$ may appear under general covariant transformations

$$\overline{c}'^{\nu_{k+2}\cdots\nu_{n-1}}=\det(\frac{\partial x^\alpha}{\partial x'^\beta})\frac{\partial x'^{\nu_{k+2}}}{\partial x^{\mu_{k+2}}}\cdots\frac{\partial x'^{\nu_{n-1}}}{\partial x^{\mu_{n-1}}}\overline{c}^{\mu_{k+2}\cdots\mu_{n-1}}$$

of a chain Φ (59). In particular, Φ contains the summand

$$\sum_{k_1+\cdots+k_i+3i=N+2} F_{\nu^1_{k_1+2}\cdots\nu^1_{n-1};\ldots;\nu^i_{k_i+2}\cdots\nu^i_{n-1}}\overline{c}'^{\nu^1_{k_1+2}\cdots\nu^1_{n-1}}\cdots\overline{c}'^{\nu^i_{k_i+2}\cdots\nu^i_{n-1}},$$

which must vanish if Φ is a cycle. This takes place only if Φ factorizes through the graded densities $\Delta^{\mu_{k+2}\cdots\mu_{n-1}}$ (54) in accordance with the expression (57).

Following the proof of Lemma 14, one can show that any $(N+2)$-cycle $\Phi\in\mathcal{P}^{0,n}_\infty\{N\}_{N+2}$ up to a boundary takes the form

$$\Phi=\sum_{0\leq|\Lambda|} G^\Lambda_{\mu_{N+2}\cdots\mu_{n-1}}\Delta^{\mu_{N+2}\cdots\mu_{n-1}}\omega, \tag{60}$$

i.e., the homology $H_2(\delta_N)$ of the complex (56) is finitely generated by the cycles $\Delta^{\mu_{N+2}\cdots\mu_{n-1}}$. Thus, the complex (56) admits the $(N+2)$-exact extension (45).

The iteration procedure is prolonged till $N=n-3$. Given the BGDA $\mathcal{P}^*\{n-3\}$, the corresponding $(n-2)$-exact complex (56) has the following $(n-1)$-exact extension. Let us consider the BGDA $\mathcal{P}^*\{n-2\}$, where $V_{n-2}=X\times\mathbb{R}$. It possesses the local bases

$$\{A, B_{\mu_1\ldots\mu_{n-1}}, \overline{s}, \overline{s}^{\mu_1\cdots\mu_{n-1}}, \overline{c}^{\mu_2\cdots\mu_{n-1}},\ldots,\overline{c}^{\mu_{n-1}},\overline{c}\},$$

where $[\overline{c}]=(n-1)\bmod 2$ and $\mathrm{Ant}[\overline{c}]=n+1$. It is provided with the nilpotent graded derivation

$$\delta_{n-2}=\delta_0+\sum_{1\leq k\leq n-3}\frac{\overleftarrow{\partial}}{\partial\overline{c}^{\mu_{k+2}\cdots\mu_{n-1}}}\Delta^{\mu_{k+2}\cdots\mu_{n-1}}+\frac{\overleftarrow{\partial}}{\partial\overline{c}}\Delta, \qquad \Delta=d_{\mu_{n-1}}\overline{c}^{\mu_{n-1}} \tag{61}$$

Then the above mentioned $(n-1)$-exact complex is

$$\begin{aligned}0\leftarrow \mathrm{Im}\,\overline{\delta}\xleftarrow{\overline{\delta}}\mathcal{P}^{0,n}_\infty[\overline{Y}^*;Y]_1\xleftarrow{\delta_0}\mathcal{P}^{0,n}_\infty\{0\}_2\xleftarrow{\delta_1}\mathcal{P}^{0,n}_\infty\{1\}_3\cdots\\ \xleftarrow{\delta_{n-3}}\mathcal{P}^{0,n}_\infty\{n-3\}_{n-1}\xleftarrow{\delta_{n-2}}\mathcal{P}^{0,n}_\infty\{n-2\}_n\xleftarrow{\delta_{n-2}}\mathcal{P}^{0,n}_\infty\{n-2\}_{n+1}.\end{aligned} \tag{62}$$

Following the proof of Lemma 14, one can show that the n-homology regularity condition is satisfied. Therefore, any n-cycle up to a δ_{n-3}-boundary takes the form

$$\Phi=\sum_{0\leq|\Lambda|}G^\Lambda\overline{c}_\Lambda.$$

The cycle condition reads

$$\delta_{n-2}\Phi=\sum_{0\leq|\Lambda|}G^\Lambda d_\Lambda\Delta=0.$$

It follows that $G^\Lambda=0$ and, consequently, $\Phi=0$. Thus, the n-homology of the complex (62) is trivial, and this complex is exact. It is a desired Koszul–Tate complex of a Lagrangian system in question. The nilpotency conditions of its boundary operator (61) restarts all the Noether identities of this Lagrangian system.

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